In class the confusion concerning basic logic arose.
Let me briefly recall the knowledge.

① How to negate a statement with quantifier:

Rule: $\neg(\forall x)(P(x)) = (\exists x)(\neg P(x))$.

$\neg(\exists x)(P(x)) = (\forall x)(\neg P(x))$.

Example: P = Prime factorization theorem (human language)

$= (\forall n \in \mathbb{Z}_+)$ (n can be written as a product of prime numbers)

In human words, every positive integer can be written as a product of prime integers.

$\neg P = (\exists n \in \mathbb{Z}_+)$ (n cannot be written as a product of prime numbers)

In human words: there exists a positive integer that cannot be be written as a product of prime numbers.

Example: P = Prime Factorization Theorem (weak)

$= (\forall n \in \mathbb{Z}_+)(\exists k \in \mathbb{Z}_+)(\exists p_1, \cdots, p_k \in \mathbb{Z}_+$ prime numbers$)$
$(n = p_1 p_2 \cdots\cdots p_k)$

In words, for every positive integer n,
there exists a positive integer k
exists k prime numbers $p_1, p_2, \cdots, p_k$

such that n can be written as the product of these k prime numbers

aka. $n = p_1 p_2 \cdots p_k$.

$$\neg P = (\exists n \in \mathbb{Z}_+)(\forall k \in \mathbb{Z}_+)(\forall p_1, \cdots, p_k \in \mathbb{Z}_+ \text{ prime numbers})$$
$$(n \neq p_1 p_2 \cdots p_k)$$

In words: there exists a positive integer n such that for every positive integer k for any k prime numbers $p_1, \cdots, p_k$ n cannot be written as a product of these prime numbers.

aka. $n \neq p_1 p_2 \cdots p_k$.

② Argue by contradiction:

Rule: $P \Rightarrow Q$ is equivalent to $\neg Q \Rightarrow \neg P$.

Reason: $P \Rightarrow Q = \neg P \vee Q$

$\neg Q \Rightarrow \neg P = \neg(\neg Q) \vee \neg P = Q \vee \neg P$.

Example: Prove by contradiction that $x^2 = 144 \Rightarrow x \neq 5$.

$\neg Q$: $(x = 5)$.

$x = 5 \Rightarrow x^2 = 25$.

$25 \neq 144 \Rightarrow x^2 \neq 144 \Rightarrow \neg P$ $\square$

Example: $(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})\left((\forall \varepsilon > 0)(a < b + \varepsilon) \Rightarrow a \leq b\right)$

Argue by contradiction:

$P(a,b) = (\forall \varepsilon > 0)(a < b + \varepsilon)$. $\neg P(a,b) = (\exists \varepsilon > 0)(a \geq b + \varepsilon)$

$Q(a,b) = (a \leq b)$ $\neg Q(a,b) = a > b$.

So $(\neg Q(a,b) \Rightarrow \neg P(a,b))$ means $((a > b) \Rightarrow (\exists \varepsilon > 0)(a \geq b + \varepsilon))$

Pf: For arbitrarily chosen $a, b \in \mathbb{R}$

$a > b \Rightarrow a - b > 0$.

I chose an ε for arbitrarily chosen a, b!

Let $\varepsilon = a - b$, then $b + \varepsilon = a$. $\Rightarrow a \geq b + \varepsilon$

i.e. for $\varepsilon = a - b$, $a \geq b + \varepsilon$.

$\Rightarrow \exists \varepsilon > 0,\ a \geq b + \varepsilon$.

Since a, b are arbitrarily chosen, we got

$(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})((a > b) \Rightarrow (\exists \varepsilon > 0,\ a > b + \varepsilon))$

Argue by deduction:

Pf: For arbitrarily chosen $a, b \in \mathbb{R}$

$\forall \varepsilon > 0,\ a < b + \varepsilon \Rightarrow a - b < \varepsilon$

So $a - b$ is strictly less than any positive number

$\Rightarrow a - b$ is a lower bound of the set of positive numbers.

$\Rightarrow a - b \leq \inf\{\varepsilon \in \mathbb{R} : \varepsilon > 0\} = 0$

$\Rightarrow a \leq b$.

Since a, b are arbitrarily chosen, we got

$(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})((\forall \varepsilon > 0)(a < b + \varepsilon) \Rightarrow a \leq b)$

Warning: You are not supposed to pick random a, b's to argue.

Other Comments :

* Let X be a subset of $\mathbb{R}$.

<u>Def</u>. $a \in \mathbb{R}$ is an upper bound of X if

$$\forall x \in X, \ x \leq a.$$

$a \in \mathbb{R}$ is the maximum of X (formally, $a = \max X$)

$$(a \in X) \wedge (\forall x \in X, \ x \leq a).$$

$a \in \mathbb{R}$ is the least upper bound of X (formally, $a = \sup X$)

$$(\forall x \in X, x \leq a) \wedge \big((\forall a' \in \mathbb{R})(a' \geq x, \forall x \in A) \Rightarrow (a' \geq a)\big)$$

Key distinction: $a = \max X \Rightarrow a = \sup X$

$a = \sup X \not\Rightarrow a = \max X$

b/c $\sup X$ does not necessarily lie in X

Useful equivalence: $a = \sup X \Leftrightarrow (\forall x \in X, a \geq x) \wedge (\forall \varepsilon > 0, \exists x \in X, x > a + \varepsilon)$.

* Axiom of completeness: $\forall X \subseteq \mathbb{R}$ bounded above $\Rightarrow \exists a \in \mathbb{R}, a = \sup X$.

Nested Interval Property:

$\forall \{ I_n = [a_n, b_n] : n = 1, 2, \cdots \}$ satisfying $(\forall m \in \mathbb{Z}_+)(I_m \supseteq I_{m+1})$.

then $\bigcap_{n=1}^{\infty} I_n \neq \phi$.

The book showed that AoC $\Rightarrow$ NIP. In fact it's also true that NIP $\Rightarrow$ AoC.

This will be seen in Chapter 2.